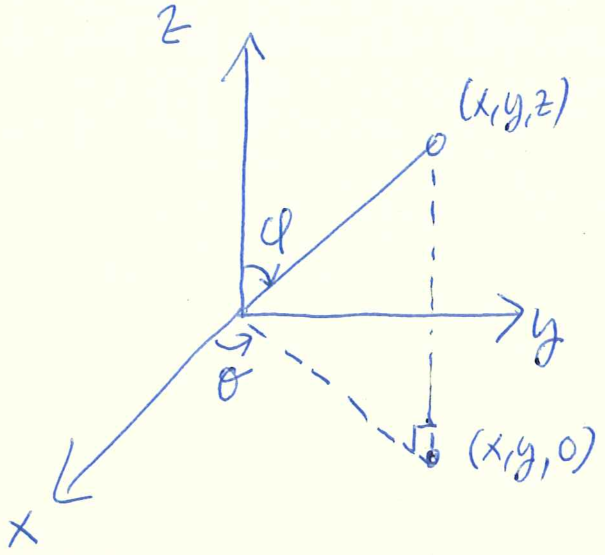
2020 A Adv. Cal. II



A point $P(x,y,z)$ can be described by (ρ,φ,θ) where

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi,$$
$$\rho \geq 0,\ \varphi \in [0,\pi],\ \theta \in [0,2\pi].$$

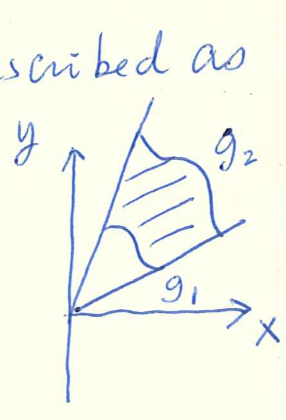
(ρ,φ,θ) is called the spherical coordinates of P.

Note that $(\rho,\theta,\varphi) \mapsto (x,y,z)$ as a mapping from $[0,\infty)\times[0,\pi]\times[0,2\pi]$ to $\mathbb{R}^3$ is onto. When restricted to $(0,\infty)\times[0,\pi]\times[0,2\pi)$ it is 1-1 onto $\mathbb{R}^3$ except the origin. (ρ,φ,θ) can be recovered by (x,y,z) via

$$\rho = \sqrt{x^2+y^2+z^2},$$
$$\varphi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},$$
$$\theta = \tan^{-1}\frac{y}{x}.$$

Recall that for a region $D \subset \mathbb{R}^2$ which is described as

$$\left\{(x,y) : \begin{array}{l} g_1(\theta) \leq r \leq g_2(\theta) \\ \theta_1 \leq \theta \leq \theta_2 \end{array}\right\},$$



$$\iint_D f = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Now, for a region Ω in $\mathbb{R}^3$ described as

$$\left\{(x,y,z): \begin{array}{l} \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta) \\ \varphi_1 \le \varphi \le \varphi_2,\ \theta_1 \le \theta \le \theta_2 \end{array}\right\},$$

we have

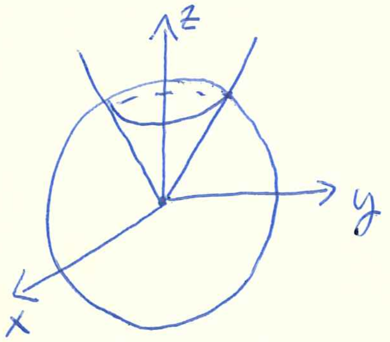
$$\iiint_\Omega f = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} \hat{f}(\rho,\varphi,\theta)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta,$$

where

$$\hat{f}(\rho,\varphi,\theta) \equiv f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

(Just have to memorize this formula, will discuss its proof later)

e.g. Let Ω be the ice cream cone bounded above by $x^2+y^2+z^2=4$ and below by $z=\sqrt{x^2+y^2}$. Express its volume and moment of inertia w.r.t. z-axis in spherical coordinates.



In spherical coordinate,

$x^2+y^2+z^2=4$ is $\rho^2=4$, $\rho=2$,

$z=\sqrt{x^2+y^2}$ is $\rho\cos\varphi = \sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$,

ie $\rho\cos\varphi = \rho\sin\varphi$, ie

$\tan\varphi = 1$, or $\varphi = \pi/4$.

So Ω is

$0 \le \rho \le 2$,
$0 \le \varphi \le \pi/4$,
$0 \le \theta \le 2\pi$.

For any ray emitting from the origin with (φ,θ) in this range, the ray hits $\rho=2$ and then leaves Ω. For rays not in this

(φ,θ) - range, it never hits Ω. Hence, the ice-cream cone is described as Ω.

$$\text{volume} = \iiint_{\Omega} 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 1\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{8}{3}\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \frac{8}{3}(-\cos\varphi)\Big|_0^{\pi/4} d\theta$$

$$= \frac{8}{3}\left(1-\frac{\sqrt{2}}{2}\right)2\pi \quad \#$$

Next,

$$I_z = \iiint_{\Omega} (x^2+y^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \left[(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2\right] \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^4 \sin^3\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{32}{5}\int_0^{2\pi}\int_0^{\pi/4} \sin^3\varphi\, d\varphi\, d\theta \qquad t=\cos\varphi$$

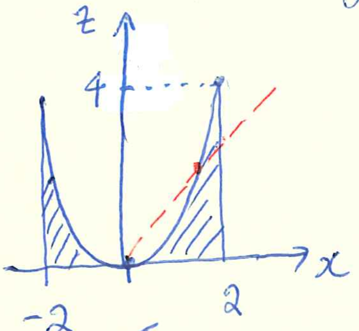
$$= \frac{32}{5}\int_0^{2\pi}\int_{\frac{\sqrt{2}}{2}}^{1} (1-t^2)\, dt\, d\theta,$$

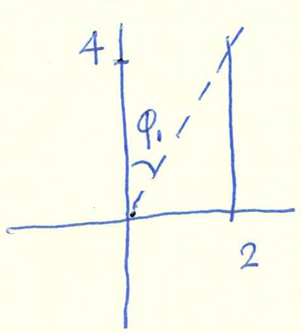
$$= \frac{32}{5}\times 2\pi \times \left(t-\frac{t^3}{3}\right)\Big|_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \frac{32}{5}\times 2\pi\times\left(\frac{2}{3}-\frac{\sqrt{2}}{4}\right)$$

$$= (8-3\sqrt{2})\frac{16\pi}{15} \quad \#$$

e.g. Let Ω be the solid under the surface $z = x^2+y^2$ over the disk $x^2+y^2 \leq 4$. Find its volume.





A ray with $\varphi \in [\varphi_1, \pi/2]$, $\theta \in [0, 2\pi]$, hits $z = x^2+y^2$ and then the cylinder $x^2+y^2=4$

Here $\tan\varphi_1 = 2/4$, ie $\varphi_1 = \tan^{-1}\frac{1}{2}$.

$z = x^2+y^2$, $\rho\cos\varphi = (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2$;

$\rho\cos\varphi = \rho^2\sin^2\varphi$, $\rho = \dfrac{\cos\varphi}{\sin^2\varphi}$

$\therefore \rho_1(\varphi,\theta) = \dfrac{\cos\varphi}{\sin^2\varphi}$

$x^2+y^2=4$, $(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 = 1$,

$\rho^2\sin^2\varphi = 1$, $\rho = 1/\sin\varphi$,

$\therefore \rho_2(\varphi,\theta) = \csc\varphi$.

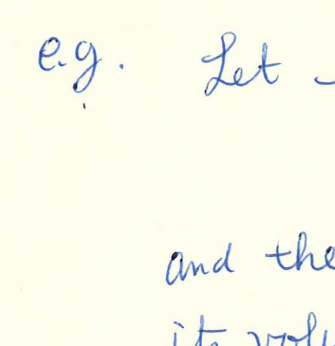
$$\therefore \text{Vol} = \iiint_{\Omega} dV = \int_0^{2\pi}\int_{\varphi_1}^{\pi/2}\int_{\frac{\cos\varphi}{\sin^2\varphi}}^{\csc\varphi} \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$$

$= \cdots$ #

e.g. Let Ω be the solid pinched between

$$x^2+y^2+(z-\tfrac{1}{2})^2 = \tfrac{1}{4}$$

and the hemisphere $x^2+y^2+z^2=4$, $z \geq 0$. Find its volume.



A ray with $\varphi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$, hits $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ and then the hemisphere.

$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$, ie

$(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi - \frac{1}{2})^2 = \frac{1}{4}$,

$$\rho^2\sin^2\varphi + \rho^2\cos^2\varphi - \rho\cos\varphi + \frac{1}{4} = \frac{1}{4}$$

$$\rho = \cos\varphi$$

$$\therefore \quad \rho_1(\theta,\varphi) = \cos\varphi$$

clearly, $\rho_2(\theta,\varphi) = 2$.

$$\text{vol} = \iiint_\Omega dV = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\varphi}^{2} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{3}(8-\cos^3\varphi)\sin\varphi \, d\varphi\, d\theta$$

$$\vdots$$

$$= \frac{31\pi}{6} \quad \#$$

A point $P(x,y,z)$ can be described by (r,θ,z)

where
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z,$$

ie, (r,θ) is the polar coordinates of (x,y). Then (r,θ,z) is called the cylindrical coordinates of P.

When Ω can be described as

$$g_1(r,\theta) \le z \le g_2(r,\theta)$$
$$(r,\theta) \in D,$$

$$\iiint_\Omega f = \iint_D \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta.$$

e.g. Find the centroid of the solid enclosed by the cylinder $x^2+y^2=4$ bounded above by the paraboloid $z=x^2+y^2$, and bounded below by the xy-plane.

By symmetry it suffices to calculate M and M_{xy}

$z=x^2+y^2$, $z=r^2$, in polar, so

Ω is $0 \le z \le r^2$

$(r,\theta) \in D_2$

D_2 is $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

$$M = \iiint_\Omega dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} 1\,dz\,r\,dr\,d\theta$$
$$\vdots$$
$$= 8\pi.$$

$$M_{xy} = \iiint_\Omega z\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta$$
$$\vdots$$
$$= \frac{32\pi}{3}.$$

$\therefore$ Centroid $= (\bar{x}, \bar{y}, \bar{z})$

$$= \left(0, 0, \frac{\frac{32\pi}{3}}{8\pi}\right)$$

$$= \left(0, 0, \frac{4}{3}\right).$$

(Con'td)

So far, we have learnt 3 ways to express a region Ω in space:

① rectangular coordinates:

$$g_1(x,y) \le z \le g_2(x,y)$$
$$(x,y) \in D,$$

where D can be described such as

$$f_1(x) \le y \le f_2(x)$$
$$a \le x \le b .$$

$$\iiint_\Omega f = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\, dy\, dx .$$

② cylindrical coordinates:

$$h_1(r,\theta) \le z \le h_2(r,\theta)$$
$$g_1(\theta) \le r \le g_2(\theta)$$
$$\theta_1 \le \theta \le \theta_2 .$$

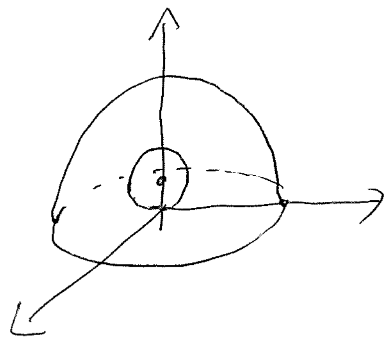
$$\iiint_\Omega f = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta .$$

③ spherical coordinates:

$$\rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta)$$
$$\varphi_1 \le \varphi \le \varphi_2$$
$$\theta_1 \le \theta \le \theta_2$$

$$\iiint_\Omega f = \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta .$$

e.g. Let Ω be the region bounded between the upper hemisphere $x^2+y^2+z^2=4$, $z \geq 0$, and the sphere $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.



Last time we know Ω can be described in spherical coordinate as

$$\cos\varphi \leq \rho \leq 2,$$
$$0 \leq \varphi \leq \pi/2$$
$$0 \leq \theta \leq 2\pi$$

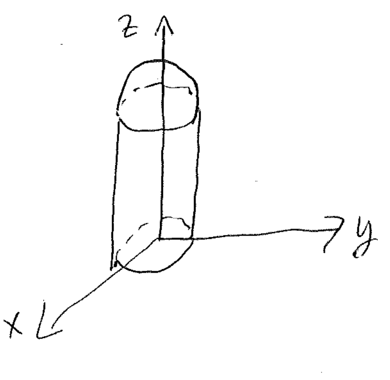
$$\therefore \iiint_\Omega f = \int_0^{2\pi} \int_0^{\pi/2} \int_{\cos\varphi}^{2} f(\cdots)\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

On the other hand, Ω can't be expressed as

$$g_1(x,y) \leq z \leq g_2(x,y) \quad \text{or}$$
$$h_1(r,\theta) \leq z \leq h_2(r,\theta)$$

without decomposing it. The rectangular and cylindrical coordinates are not suitable.

e.g. Let Ω be the bullet-shaped solid bounded above by the upper side of $x^2+y^2+(z-2)^2=1$, and the cylinder $x^2+y^2=1$ on the side, over the xy-plane.



$$x^2+y^2+(z-2)^2=1$$
$$z = 2+\sqrt{1-x^2-y^2}.$$

In rectsl coordinate

$$0 \le z \le 2+\sqrt{1-x^2-y^2}$$

$$(x,y) \in D_1$$

$$D_1 : -1 \le x \le 1$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1+x^2}$$

$$\therefore \iiint_\Omega f = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} \int_0^{2+\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx.$$

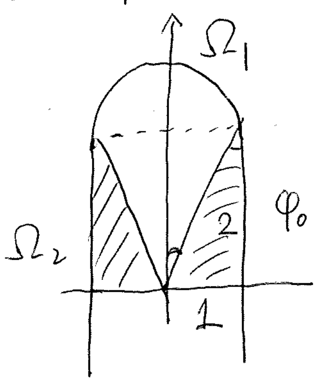
In cylinderical coor.

$$D_1 : \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$\iiint_\Omega f = \int_0^{2\pi} \int_0^{1} \int_0^{2+\sqrt{1-r^2}} f(\cdots)\, dz\, r dr\, d\theta.$$

In spherical coordinats $\Omega = \Omega_1 \cup \Omega_2$.



The upper hemisphere : $x^2+y^2+(z-2)^2=1$

$$\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta + (\rho\cos\varphi - 2)^2 = 1$$

$$\rho^2 - 4\cos\varphi\, \rho + 3 = 0$$

$$\rho = \frac{4\cos\varphi + \sqrt{16\cos^2\varphi + 12}}{2}$$

$$\rho = 2\cos\varphi + \sqrt{4\cos^2\varphi + 3}$$

$$\Omega_1 : \quad 0 \le \rho \le 2\cos\varphi + \sqrt{4\cos^2\varphi + 3}$$

$$0 \le \varphi \le \varphi_0$$

$$0 \le \theta \le 2\pi.$$

$\tan\varphi_0 = \frac{1}{2} \quad \therefore \quad \varphi_0 = \tan^{-1}\frac{1}{2}$.

The cylinder $x^2+y^2=1$, $(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2=1$

$\therefore \quad \rho = 1/\sin\varphi = \csc\varphi$.

$$\Omega_2: \quad 0 \le \rho \le \csc\varphi$$
$$\varphi_0 \le \varphi \le \pi/2$$
$$0 \le \theta \le 2\pi$$

$$\iiint_{\Omega} f = \iiint_{\Omega_1} f + \iiint_{\Omega_2} f, \quad \text{where}$$

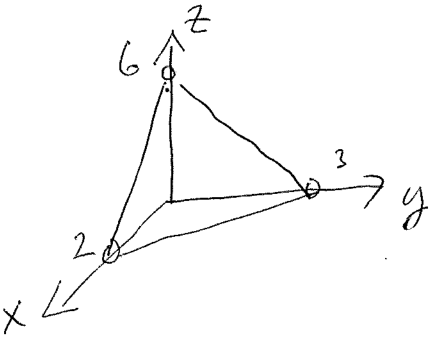
$$\iiint_{\Omega_1} f = \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{2\cos\varphi+\sqrt{4\cos^2\varphi+3}} f(\cdots)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

and

$$\iiint_{\Omega_2} f = \int_0^{2\pi}\int_{\varphi_0}^{\pi/2}\int_0^{\csc\varphi} f(\cdots)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

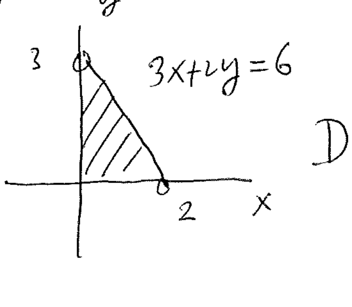
Clearly, the expression in cylindrical coordinate is the simplest one.

e.g. Let Ω be the wedge in the first octant formed by $3x+2y+z=6$.



In rectangular coordinates,

$$0 \le z \le 6-3x-2y$$
$$(x,y) \in D,$$



$$\iiint_\Omega f = \int_0^2 \int_0^{\frac{1}{2}(6-3x)} \int_0^{6-3x-2y} f(\cdots)\,dz\,dy\,dx .$$

In cylinderical coordinate, D is expressed as

$$3x+2y=6, \quad 3r\cos\theta + 2r\sin\theta = 6,$$

$$r = \frac{6}{3\cos\theta + 2\sin\theta}$$

$$0 \le r \le \frac{6}{3\cos\theta + 2\sin\theta},$$

$$0 \le \theta \le \pi/2$$

$$\therefore \iiint_\Omega f = \int_0^{\pi/2} \int_0^{\frac{6}{3\cos\theta+2\sin\theta}} \int_0^{6-3r\cos\theta-2r\sin\theta} f(\cdots)\,dz\,r\,dr\,d\theta .$$

In spherical coor.

$$3x+2y+z=6,$$

$$3\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + \rho\cos\varphi = 6$$

$$\rho = \frac{6}{3\sin\varphi\cos\theta + 2\sin\varphi\sin\theta + \cos\varphi} .$$

$$\Omega : \quad 0 \le \rho \le \frac{6}{3\sin\varphi\cos\theta + 2\sin\varphi\sin\theta + \cos\varphi}$$

$$0 \le \varphi \le \pi/2$$

$$0 \le \theta \le \pi/2$$

$$\iiint_\Omega f = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{6}{3\sin\varphi\cos\theta+2\sin\varphi\sin\theta+\cos\varphi}} f(\cdots)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta .$$

Clearly, the rectgular coor. is best suited in this example.